

No part of this product may be reproduced in any form or by any electronic or mechanical means, including information storage and retrieval systems, without written permission from the IB.

Additionally, the license tied with this product prohibits commercial use of any selected files or extracts from this product. Use by third parties, including but not limited to publishers, private teachers, tutoring or study services, preparatory schools, vendors operating curriculum mapping services or teacher resource digital platforms and app developers, is not permitted and is subject to the IB's prior written consent via a license. More information on how to request a license can be obtained from <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

Aucune partie de ce produit ne peut être reproduite sous quelque forme ni par quelque moyen que ce soit, électronique ou mécanique, y compris des systèmes de stockage et de récupération d'informations, sans l'autorisation écrite de l'IB.

De plus, la licence associée à ce produit interdit toute utilisation commerciale de tout fichier ou extrait sélectionné dans ce produit. L'utilisation par des tiers, y compris, sans toutefois s'y limiter, des éditeurs, des professeurs particuliers, des services de tutorat ou d'aide aux études, des établissements de préparation à l'enseignement supérieur, des fournisseurs de services de planification des programmes d'études, des gestionnaires de plateformes pédagogiques en ligne, et des développeurs d'applications, n'est pas autorisée et est soumise au consentement écrit préalable de l'IB par l'intermédiaire d'une licence. Pour plus d'informations sur la procédure à suivre pour demander une licence, rendez-vous à l'adresse suivante : <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

No se podrá reproducir ninguna parte de este producto de ninguna forma ni por ningún medio electrónico o mecánico, incluidos los sistemas de almacenamiento y recuperación de información, sin que medie la autorización escrita del IB.

Además, la licencia vinculada a este producto prohíbe el uso con fines comerciales de todo archivo o fragmento seleccionado de este producto. El uso por parte de terceros —lo que incluye, a título enunciativo, editoriales, profesores particulares, servicios de apoyo académico o ayuda para el estudio, colegios preparatorios, desarrolladores de aplicaciones y entidades que presten servicios de planificación curricular u ofrezcan recursos para docentes mediante plataformas digitales— no está permitido y estará sujeto al otorgamiento previo de una licencia escrita por parte del IB. En este enlace encontrará más información sobre cómo solicitar una licencia: <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

Further mathematics
Higher level
Paper 1

Friday 23 October 2020 (afternoon)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 6]

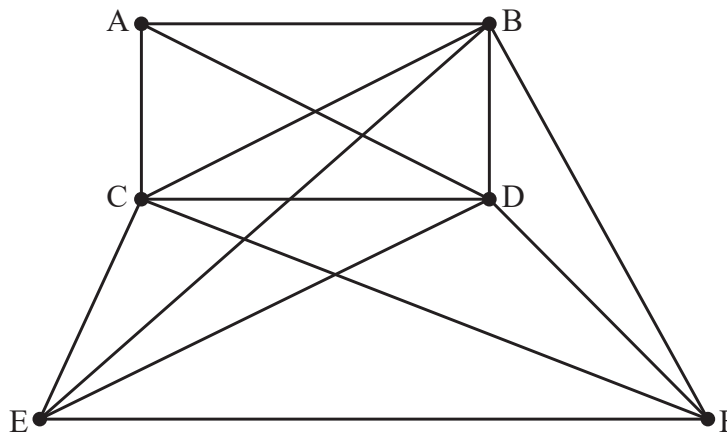
Use l'Hôpital's rule to determine the value of

$$\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} . \quad [6]$$

2. [Maximum mark: 8]

The following diagram shows the graph G .

diagram not to scale



- (a) Verify that G satisfies the handshaking lemma. [3]
- (b) Show that G cannot be redrawn as a planar graph. [3]
- (c) State, giving a reason, whether G contains an Eulerian circuit. [2]

3. [Maximum mark: 8]

The binary operation $*$ is defined on the set $S = \{a, b, c, d, e, f\}$ by the following Cayley table.

$*$	a	b	c	d	e	f
a	c	e	a	f	d	b
b	d	c	b	e	f	a
c	a	b	c	d	e	f
d	b	f	d	c	a	e
e	f	a	e	b	c	d
f	e	d	f	a	b	c

- (a) Explain why this table is a Latin square. [1]
- (b) State the identity element. [1]
- (c) Determine the inverse of each element of S . [1]
- (d) Find
 - (i) $a * (b * d)$;
 - (ii) $(a * b) * d$. [3]
- (e) State, giving a reason, whether $\{S, *\}$ is a group. [2]

4. [Maximum mark: 11]

The matrix A is given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- (a) By considering the determinant of a relevant matrix, show that the eigenvalues, λ , of A satisfy the equation

$$\lambda^2 - \alpha\lambda + \beta = 0,$$

where α and β are functions of a, b, c, d to be determined. [4]

- (b) (i) Verify that

$$A^2 - \alpha A + \beta I = \mathbf{0}.$$

- (ii) Assuming that A is non-singular, use the result in part (b)(i) to show that

$$A^{-1} = \frac{1}{\beta}(\alpha I - A). \quad [7]$$

5. [Maximum mark: 8]

The continuous random variable X has cumulative distribution function F , where $F(a) = 0$ and $F(b) = 1$.

(a) Using integration by parts, show that $E(X) = b - \int_a^b F(x)dx$. [4]

$$\text{Let } F(x) = \begin{cases} 0, & x < 0 \\ \tan x, & 0 \leq x \leq \frac{\pi}{4} \\ 1, & x > \frac{\pi}{4} \end{cases} .$$

(b) Using the result from part (a), determine $E(X)$. Give your answer correct to three significant figures. [2]

(c) Determine the median of X , giving your answer correct to three significant figures. [2]

6. [Maximum mark: 6]

Find the smallest positive value of x satisfying the following two linear congruences simultaneously.

$$\begin{aligned} 5x &\equiv 4 \pmod{11} \\ 11x &\equiv 6 \pmod{7} \end{aligned} \quad [6]$$

7. [Maximum mark: 12]

Points in the plane are subjected to a transformation T in which the point (x, y) is transformed to the point (x', y') where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

(a) Describe, in words, the effect of the transformation T . [1]

(b) (i) Show that the points $A(1, 4), B(4, 8), C(8, 5), D(5, 1)$ form a square.

(ii) Determine the area of this square.

(iii) Find the coordinates of A', B', C', D' , the points to which A, B, C, D are transformed under T .

(iv) Show that $A' B' C' D'$ is a parallelogram.

(v) Determine the area of this parallelogram. [11]

8. [Maximum mark: 12]

Consider the group $\{S, \times_{13}\}$, where $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and \times_{13} denotes multiplication modulo 13.

- (a) Find the five pairs of distinct elements of S , such that each element in a pair is the inverse of the other element in the pair. [4]
- (b) Determine the subgroup of $\{S, \times_{13}\}$
 - (i) of order 2;
 - (ii) of order 3. [3]
- (c) You are given that $\{T, \times_{13}\}$ is a subgroup of $\{S, \times_{13}\}$, where $T = \{1, 5, 8, 12\}$.
 - (i) Determine the cosets of the elements 2, 3 and 4 with respect to $\{T, \times_{13}\}$.
 - (ii) State the general result concerning the elements contained in different cosets that is verified by your answer to part (c)(i). [5]

9. [Maximum mark: 13]

The discrete random variable X has probability distribution

$$P(X=x) = pq^x, x \in \mathbb{N}, 0 < p < 1, q = 1 - p.$$

- (a) (i) Show that the probability generating function of X is given by

$$G_x(t) = \frac{p}{1-qt}.$$

- (ii) Hence find $\text{Var}(X)$ in terms of p . Express your answer in its simplest form. [9]
- (b) The random variable Y is defined by

$$Y = X_1 + X_2 + X_3 + X_4$$

where X_1, X_2, X_3, X_4 is a random sample from the distribution of X .

- (i) Write down the probability generating function of Y .
- (ii) Hence determine an expression for $P(Y=3)$ in terms of p . [4]

10. [Maximum mark: 7]

The matrix M is given by

$$M = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 2 & 2 & 3 \\ -1 & 4 & 0 & 5 \\ 1 & 7 & 1 & 9 \end{bmatrix}.$$

(a) Justifying your answer, determine the rank of M . [3]

Let the set $S = \left\{ \begin{bmatrix} 2 \\ 5 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \\ 9 \end{bmatrix} \right\}$, that is the four columns of M .

(b) Give a reason why S does not span the space of four-dimensional column vectors. [1]

(c) Determine whether the vector $\begin{bmatrix} 7 \\ 12 \\ 2 \\ 9 \end{bmatrix}$ belongs to the subspace spanned by S . [3]

11. [Maximum mark: 12]

(a) Use the integral test to show that the infinite series

$$S = \sum_{n=2}^{\infty} \frac{\ln n}{n^2} \text{ is convergent.} \quad [8]$$

(b) (i) Sketch the graph of $y = \frac{\ln x}{x^2}$ for $x \geq 2$.

(ii) Hence by considering appropriate Riemann sums, show that an upper bound for S is $\frac{1}{2} + \frac{3}{4} \ln 2$. [4]

12. [Maximum mark: 6]

The points D, E, F lie on the sides [BC], [CA], [AB], respectively, of a triangle ABC. The segments [AD], [BE], [CF] meet at O. Given that [FE] is parallel to [BC], show that $BD = CD$.

[6]

13. [Maximum mark: 10]

Observations on 12 pairs of values of the random variables X, Y yielded the following results.

$$\Sigma x = 76.3, \Sigma x^2 = 563.7, \Sigma y = 72.2, \Sigma y^2 = 460.1, \Sigma xy = 495.4$$

(a) (i) Calculate the value of r , the product moment correlation coefficient of the sample.

(ii) Assuming that the distribution of X, Y is bivariate normal with product moment correlation coefficient ρ , calculate the p -value of your result when testing the hypotheses $H_0: \rho = 0; H_1: \rho > 0$.

(iii) State whether your p -value suggests that X and Y are independent.

[7]

(b) Given a further value $x = 5.2$ from the distribution of X, Y , predict the corresponding value of y . Give your answer to one decimal place.

[3]

14. [Maximum mark: 11]

In the triangle ABC, $AB = 8, BC = 12$ and $AC = 10$. A circle is inscribed in this triangle.

(a) Find the lengths of the tangents from A, B and C to this inscribed circle.

[3]

(b) (i) Show that the area of the triangle ABC is $15r$, where r denotes the radius of the inscribed circle.

(ii) Show that $\sin \hat{A} = \frac{3\sqrt{7}}{8}$.

(iii) Using parts (b) (i) and (ii), or otherwise, show that r is equal to \sqrt{N} , where N is a positive integer whose value is to be determined.

[8]

15. [Maximum mark: 7]

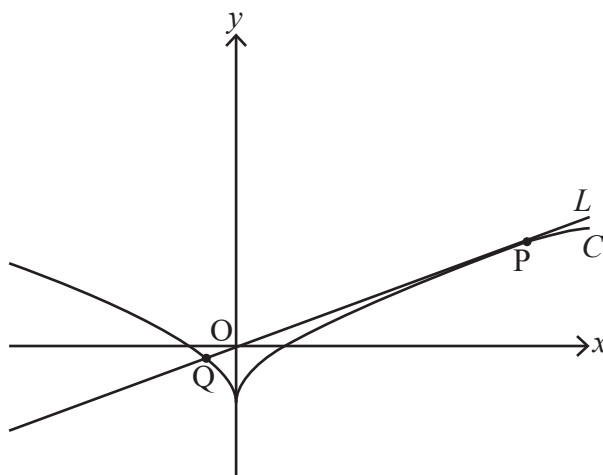
Let $(1021)_n$ denote a number expressed in number base n .

Use mathematical induction to prove that $(1021)_n$ is not divisible by 3, for $n \geq 3$.

[7]

16. [Maximum mark: 13]

The following diagram shows part of the curve C with parametric equations $x = t^3$, $y = t^2 - 1$, $t \in \mathbb{R}$.



The line L passes through the origin O and is tangential to C at the point $P(p^3, p^2 - 1)$, where $p > 0$. The line L intersects C again at the point Q .

(a) Determine

(i) the equation of L , giving the gradient in its exact form.

(ii) the exact coordinates of P .

[8]

(b) Determine the exact coordinates of Q .

[5]