Diploma Programme
Programme du diplôme
Programa del Diploma

Further mathematics
Higher level
Paper 1

Friday 23 October 2020 (afternoon)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 6]

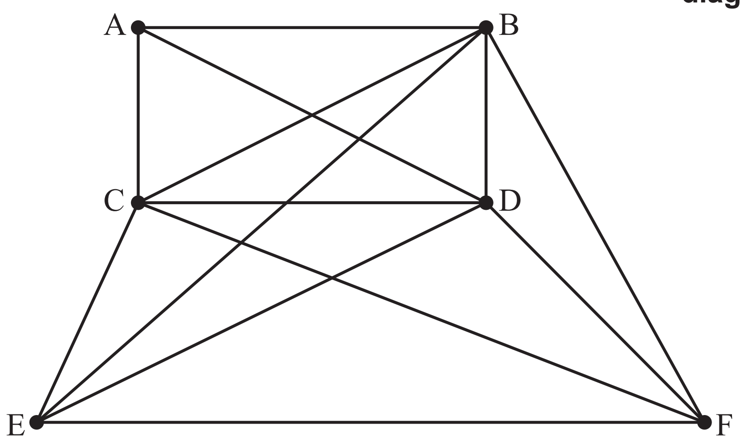
Use l'Hôpital's rule to determine the value of

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}.$$ [6]

2. [Maximum mark: 8]

The following diagram shows the graph G.

diagram not to scale



(a) Verify that G satisfies the handshaking lemma. [3]

(b) Show that G cannot be redrawn as a planar graph. [3]

(c) State, giving a reason, whether G contains an Eulerian circuit. [2]

3. [Maximum mark: 8]

The binary operation $*$ is defined on the set $S = \{a, b, c, d, e, f\}$ by the following Cayley table.

$*$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$
$\boldsymbol{a}$	c	e	a	f	d	b
$\boldsymbol{b}$	d	c	b	e	f	a
$\boldsymbol{c}$	a	b	c	d	e	f
$\boldsymbol{d}$	b	f	d	c	a	e
$\boldsymbol{e}$	f	a	e	b	c	d
$\boldsymbol{f}$	e	d	f	a	b	c

(a) Explain why this table is a Latin square. [1]

(b) State the identity element. [1]

(c) Determine the inverse of each element of S. [1]

(d) Find

(i) $a * (b * d)$;

(ii) $(a * b) * d$. [3]

(e) State, giving a reason, whether $\{S, *\}$ is a group. [2]

4. [Maximum mark: 11]

The matrix $\boldsymbol{A}$ is given by $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) By considering the determinant of a relevant matrix, show that the eigenvalues, λ, of $\boldsymbol{A}$ satisfy the equation

$$\lambda^2 - \alpha\lambda + \beta = 0,$$

where α and β are functions of a, b, c, d to be determined. [4]

(b) (i) Verify that

$$\boldsymbol{A}^2 - \alpha\boldsymbol{A} + \beta\boldsymbol{I} = \boldsymbol{0}.$$

(ii) Assuming that $\boldsymbol{A}$ is non-singular, use the result in part (b)(i) to show that

$$\boldsymbol{A}^{-1} = \frac{1}{\beta}(\alpha\boldsymbol{I} - \boldsymbol{A}).$$ [7]

5. [Maximum mark: 8]

The continuous random variable X has cumulative distribution function F, where $F(a) = 0$ and $F(b) = 1$.

(a) Using integration by parts, show that $\mathrm{E}(X) = b - \int_a^b F(x)\mathrm{d}x$. [4]

Let $F(x) = \begin{cases} 0, & x < 0 \\ \tan x, & 0 \le x \le \frac{\pi}{4} \\ 1, & x > \frac{\pi}{4} \end{cases}$.

(b) Using the result from part (a), determine $\mathrm{E}(X)$. Give your answer correct to three significant figures. [2]

(c) Determine the median of X, giving your answer correct to three significant figures. [2]

6. [Maximum mark: 6]

Find the smallest positive value of x satisfying the following two linear congruences simultaneously.

$$5x \equiv 4 \pmod{11}$$
$$11x \equiv 6 \pmod{7}$$ [6]

7. [Maximum mark: 12]

Points in the plane are subjected to a transformation T in which the point (x, y) is transformed to the point (x', y') where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) Describe, in words, the effect of the transformation T. [1]

(b) (i) Show that the points $\mathrm{A}(1, 4), \mathrm{B}(4, 8), \mathrm{C}(8, 5), \mathrm{D}(5, 1)$ form a square.

(ii) Determine the area of this square.

(iii) Find the coordinates of $\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{D}'$, the points to which $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ are transformed under T.

(iv) Show that $\mathrm{A}'\,\mathrm{B}'\,\mathrm{C}'\,\mathrm{D}'$ is a parallelogram.

(v) Determine the area of this parallelogram. [11]

8. [Maximum mark: 12]

Consider the group $\{S, \times_{13}\}$, where $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and $\times_{13}$ denotes multiplication modulo 13.

(a) Find the five pairs of distinct elements of S, such that each element in a pair is the inverse of the other element in the pair. [4]

(b) Determine the subgroup of $\{S, \times_{13}\}$

(i) of order 2;

(ii) of order 3. [3]

(c) You are given that $\{T, \times_{13}\}$ is a subgroup of $\{S, \times_{13}\}$, where $T = \{1, 5, 8, 12\}$.

(i) Determine the cosets of the elements $2, 3$ and 4 with respect to $\{T, \times_{13}\}$.

(ii) State the general result concerning the elements contained in different cosets that is verified by your answer to part (c)(i). [5]

9. [Maximum mark: 13]

The discrete random variable X has probability distribution

$$\mathrm{P}(X = x) = pq^x, x \in \mathbb{N}, 0 < p < 1, q = 1 - p.$$

(a) (i) Show that the probability generating function of X is given by

$$G_x(t) = \frac{p}{1 - qt}.$$

(ii) Hence find $\mathrm{Var}(X)$ in terms of p. Express your answer in its simplest form. [9]

(b) The random variable Y is defined by

$$Y = X_1 + X_2 + X_3 + X_4$$

where X_1, X_2, X_3, X_4 is a random sample from the distribution of X.

(i) Write down the probability generating function of Y.

(ii) Hence determine an expression for $\mathrm{P}(Y = 3)$ in terms of p. [4]

10. [Maximum mark: 7]

The matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 2 & 2 & 3 \\ -1 & 4 & 0 & 5 \\ 1 & 7 & 1 & 9 \end{bmatrix}.$$

(a) Justifying your answer, determine the rank of $\boldsymbol{M}$. [3]

Let the set $S = \left\{ \begin{bmatrix} 2 \\ 5 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \\ 9 \end{bmatrix} \right\}$, that is the four columns of $\boldsymbol{M}$.

(b) Give a reason why S does not span the space of four-dimensional column vectors. [1]

(c) Determine whether the vector $\begin{bmatrix} 7 \\ 12 \\ 2 \\ 9 \end{bmatrix}$ belongs to the subspace spanned by S. [3]

11. [Maximum mark: 12]

(a) Use the integral test to show that the infinite series

$$S = \sum_{n=2}^{\infty} \frac{\ln n}{n^2} \text{ is convergent.}$$ [8]

(b) (i) Sketch the graph of $y = \frac{\ln x}{x^2}$ for $x \geq 2$.

(ii) Hence by considering appropriate Riemann sums, show that an upper bound for S is $\frac{1}{2} + \frac{3}{4}\ln 2$. [4]

12. [Maximum mark: 6]

The points $\mathrm{D}, \mathrm{E}, \mathrm{F}$ lie on the sides $[\mathrm{BC}], [\mathrm{CA}], [\mathrm{AB}]$, respectively, of a triangle ABC. The segments $[\mathrm{AD}], [\mathrm{BE}], [\mathrm{CF}]$ meet at O. Given that $[\mathrm{FE}]$ is parallel to $[\mathrm{BC}]$, show that $\mathrm{BD} = \mathrm{CD}$. [6]

13. [Maximum mark: 10]

Observations on 12 pairs of values of the random variables X, Y yielded the following results.

$$\Sigma x = 76.3,\ \Sigma x^2 = 563.7,\ \Sigma y = 72.2,\ \Sigma y^2 = 460.1,\ \Sigma xy = 495.4$$

(a) (i) Calculate the value of r, the product moment correlation coefficient of the sample.

(ii) Assuming that the distribution of X, Y is bivariate normal with product moment correlation coefficient ρ, calculate the p-value of your result when testing the hypotheses $H_0: \rho = 0;\ H_1: \rho > 0$.

(iii) State whether your p-value suggests that X and Y are independent. [7]

(b) Given a further value $x = 5.2$ from the distribution of X, Y, predict the corresponding value of y. Give your answer to one decimal place. [3]

14. [Maximum mark: 11]

In the triangle ABC, $\mathrm{AB} = 8$, $\mathrm{BC} = 12$ and $\mathrm{AC} = 10$. A circle is inscribed in this triangle.

(a) Find the lengths of the tangents from A, B and C to this inscribed circle. [3]

(b) (i) Show that the area of the triangle ABC is $15r$, where r denotes the radius of the inscribed circle.

(ii) Show that $\sin \hat{A} = \dfrac{3\sqrt{7}}{8}$.

(iii) Using parts (b) (i) and (ii), or otherwise, show that r is equal to $\sqrt{N}$, where N is a positive integer whose value is to be determined. [8]

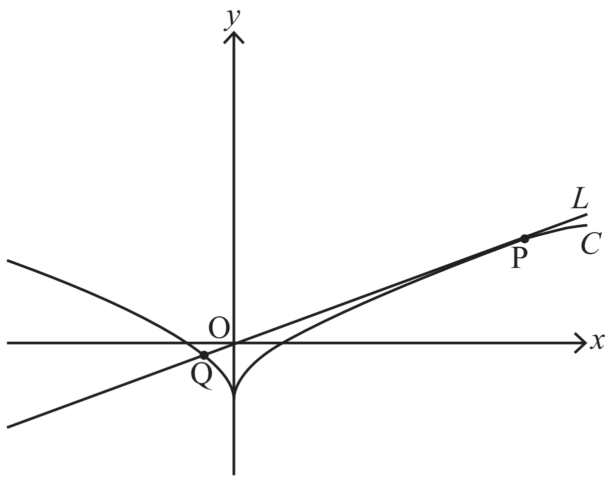
15. [Maximum mark: 7]

Let $(1021)_n$ denote a number expressed in number base n.

Use mathematical induction to prove that $(1021)_n$ is not divisible by 3, for $n \geq 3$. [7]

16. [Maximum mark: 13]

The following diagram shows part of the curve C with parametric equations $x = t^3$, $y = t^2 - 1$, $t \in \mathbb{R}$.



The line L passes through the origin O and is tangential to C at the point $\mathrm{P}(p^3, p^2 - 1)$, where $p > 0$. The line L intersects C again at the point Q.

(a) Determine

(i) the equation of L, giving the gradient in its exact form.

(ii) the exact coordinates of P. [8]

(b) Determine the exact coordinates of Q. [5]